Transmission Overhaul and Component Replacement Predictions Using Weibull and Renewal Theory

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Two methods of estimating the time between transmission overhauls and the number of replacement components needed were presented. The first method assumes replacement of all components during an overhaul of a failed transmission (full replacement method). The second method assumes replacement of failed components only (partial replacement method). Both methods assume the transmission components follow a two-parameter Weibull failure distribution. Renewal theory was presented to estimate the number of component replacements in a transmission for both methods. For the partial replacement method, renewal theory was used with the individual component life predictions to estimate the number of component replacements needed and the transmission time between overhauls. For the full replacement method, renewal theory was used with a transmission system life model to estimate the number of replacement transmissions needed and the transmission time between overhauls. Confidence statistics were applied to both methods to improve the statistical estimate of sample behavior. A transmission example was presented to illustrate use of both methods.

Nomenclature

b = Weibull slope
e = base of the natural logarithm
F = probability distribution function, probability of failure
F_k = probability of at least k failures
f = probability density function
l = life, h
M = renewal function
M_e = approximate renewal function
N_r = number of replacements
Q = sample size
R = probability of survival, $(1 - F)$
x = integration time variable, h
z_{10} = number of standard deviations from the mean which cuts off a 10% population tail
Γ = the gamma function
θ = characteristic life, h
μ_3 = third moment of a probability density function
σ = standard deviation

Subscripts
av = average or mean
f = Weibull function
i = index
k = index
me = approximate renewal function
n = number of components in system
r = replacement function
s = system
10 = 90% reliability
90 = 90% confidence

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Introduction

THE in-flight service reliability of aircraft transmissions is much greater than the design reliability of their components. Transmission overhauls provide the difference. By monitoring the onset of potential transmission fatigue failures, just-in-time overhauls maintain the transmission economically.[1,2] One cause for propulsion system overhauls is the finite fatigue life of drive system components. The two-parameter Weibull distribution describes the statistics of drive system bearing and gear life.[3-5]

Component reliabilities and lives affect transmission maintenance costs, which are significant. Estimates of these costs are important in the design stage of a transmission.[6] The two-parameter Weibull distribution provides information on component reliability and life. It does not predict overhaul frequency directly.

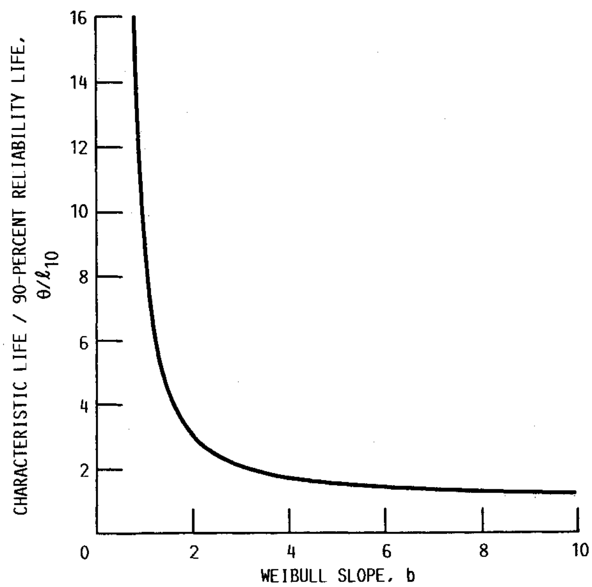

Fig. 1 Characteristic life to 90% reliability life ratio for a Weibull distribution as a function of the Weibull slope.

Two methods are presented to convert component life statistics into overhaul frequency values. The first is the transmission system life model. This model is a two-parameter Weibull distribution for the transmission system life.[7,8] The second is renewal theory.

Renewal theory is a statistical model that describes the maintenance cycle. The theory considers the ongoing sequence of use, failure onset, repair, and return to use. For this sequence, renewal theory predicts the timing of transmission or component replacement and the number of replacements needed to support the service maintenance schedule.[9-12]

Confidence theory complements these statistical methods with estimates of the likelihood of the predictions. Higher confidence levels require more spare parts to cover a greater range of possible situations.[10,11]

The purpose of this research is to provide a methodology for calculating transmission overhaul timing and the number of component replacements. It presents the theories and applies them to a simple transmission to illustrate their use. Estimates of drive system and component lives and replacement needs are essential in design. These estimates provide a comparison of the relative worth of different designs from a safety and maintenance-cost perspective. They also help assess the cost of operating a proposed drive system.

Component Life and Reliability

The two-parameter Weibull distribution is commonly used to describe fatigue life data.[13,14] It can describe a wide variety of life patterns. The reliability of a component is the complement of its probability of failure.

In statistics, reliability is a double negative. Reliability or the act of surviving is the state of not having failed. Statistics count direct events such as the act of failing. A part can fail only once. It survives for its entire life. Thus, the probability of failure is a direct statistic. The probability of failure for the two-parameter Weibull distribution is

$$F = 1.0 - e^{-(l/\theta)^b} \tag{1}$$

where F is the probability of failure expressed as a decimal, l is the component life in million load cycles or hours. The two Weibull parameters are θ and b.

The derivative of Eq. (1) with respect to life is the probability density function f:

$$f = \frac{b}{\theta}\left(\frac{l}{\theta}\right)^{b-1} e^{-(l/\theta)^b} \tag{2}$$

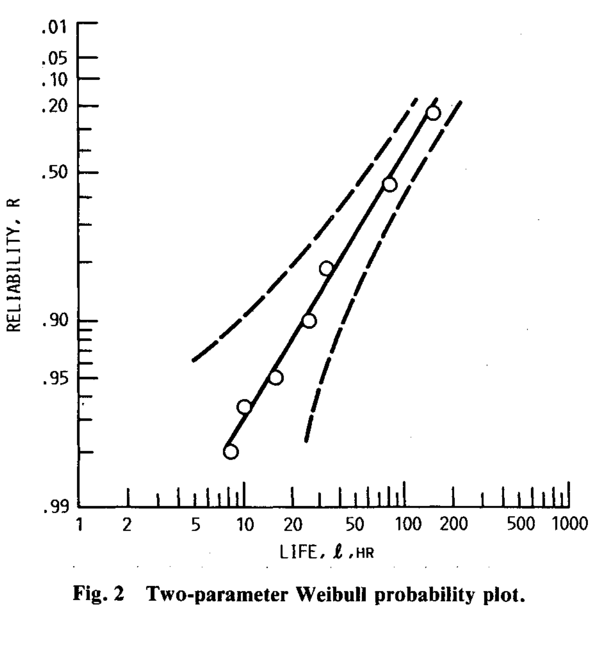

Fig. 2 Two-parameter Weibull probability plot.

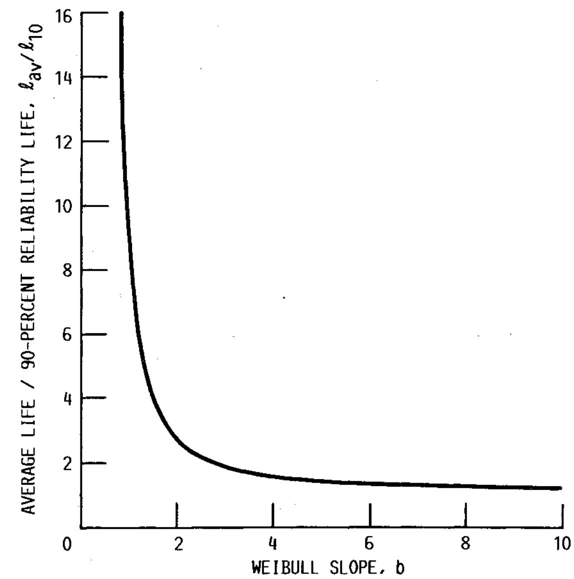

Fig. 3 Average life to 90% reliability life ratio for a Weibull distribution as a function of the Weibull slope.

The probability density function is a histogram of life failures for a unit population. It presents the scatter in the component lives.

The Weibull reliability function is often expressed as

$$\ln\left(\frac{1}{R}\right) = \left(\frac{l}{\theta}\right)^b \tag{3}$$

For a 90% probability of survival, $R = 0.9$ and $l = l_{10}$. Solving for the characteristic life gives

$$\theta = \ln\left(\frac{1}{0.9}\right)^{-1/b} l_{10} \tag{4}$$

Figure 1 is a plot of the ratio of the characteristic life to the 90% reliability life as a function of the Weibull slope. Substituting θ of Eq. (4) in Eq. (3) gives

$$\ln\left(\frac{1}{R}\right) = \ln\left(\frac{1}{0.9}\right)\left(\frac{l}{l_{10}}\right)^b \tag{5}$$

Equation (5) is the form used by manufacturers to present the two-parameter Weibull distribution characteristics of bearings.[15]

In both Eqs. (3) and (5), the logarithm of the reliability reciprocal is proportional to the life raised to the Weibull slope. Taking the logarithm of either equation generates a straight-line plot as shown in Fig. 2. The plot is a probability graph for the two-parameter Weibull distribution.

This graph aids in determining the distribution parameter values for fatigue test data.[16] The plotted test data are the results of a series of identical life tests for a sample set of identical components. The first failure determines the highest reliability data point. The next failure determines the next lowest reliability data point, and so on.

The average life is the mean time to failure (MTTF). It is the sum of all times to failure divided by the total number of the failures. The total number of failures for a continuous probability distribution is unity by definition. The sum of all times to failure is the integral of the product of time or life and the probability density function. The limits on the integral are from zero to infinity. The average life is

$$l_{av} = \text{MTTF} = \int_0^\infty l f(l)\,dl \tag{6}$$

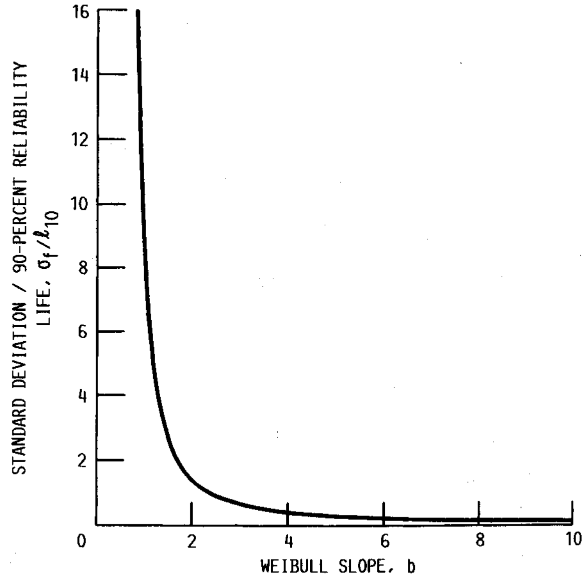

Fig. 4 Standard deviation to 90% reliability life ratio for a Weibull distribution as a function of the Weibull slope.

For a Weibull failure distribution, the solution to this integral involves the well-known gamma function Γ.[17] The solution is the gamma function multiplied by the characteristic life Θ

$$l_{av} = \mathrm{MTTF} = \Theta\Gamma\left(1 + \frac{1}{b}\right) \tag{7}$$

Figure 3 is a plot of the ratio of the average life to the 90% reliability life as a function of the Weibull slope. The average life equals the characteristic life when $b = 1$.

The standard deviation of a failure distribution is

$$\sigma_f = \left[\int_0^{\infty}(l - l_{av})^2 f(l)\mathrm{d}l\right]^{1/2} \tag{8}$$

In terms of the characteristic life, the Weibull slope, and the gamma function, the standard deviation of the two-parameter Weibull distribution is

$$\sigma_f = \Theta\left[\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right)\right]^{1/2} \tag{9}$$

The standard deviation of a distribution is a measure of the scatter of the distribution. It is valuable in estimating a confidence limit for the average life. Figure 4 is a plot of the ratio of the standard deviation to the 90% reliability life as a function of the Weibull slope. At a slope of $b = 1$, the Weibull distribution is the exponential distribution and has a large scatter. As the slope increases, the scatter decreases rapidly.

System Life and Reliability

One model for the life of a drive system is the strict series probability model.[8] This model compares a system of load-carrying gears and bearings to a chain of links. A chain fails when any single link fails. So too, a drive system requires repair when any component requires replacement or repair. In the strict series probability model, the reliability of a system R_s is the product of the reliabilities of all the components

$$R_s = \prod_{i=1}^{n} R_i \tag{10}$$

The high speed of drive system components and the scattering of loose debris warrant the strict series probability model. If any component fails, debris may be present which could damage other components. Therefore, a drive system requires an overhaul to return it to a high state of reliability once any element fails.

Taking the logarithm of the reciprocal of Eq. (10) and using Eq. (5) for each component yields

$$\ell n\left(\frac{1}{R_s}\right) = \ell n\left(\frac{1}{0.9}\right)\sum_{i=1}^{n}\left(\frac{l_s}{l_{i10}}\right)^{b_i} \tag{11}$$

In Eq. (11), l_s is the life of the entire drive system for the system reliability R_s. It is also the life of each component at the same drive system reliability R_s. For consistency in Eq. (11), all of the component lives must be defined in the same units. The unit chosen is hours.

Equation (11) is not a simple two-parameter Weibull relationship between system life and system reliability. The equation is a true two-parameter Weibull distribution only when all of the Weibull exponents b_i are equal. In general, this is not the case. Thus, R_s as a function of l_s when plotted as in Fig. 2 may produce a curve rather than a straight line. A true two-parameter Weibull distribution can be approximated quite well, however, by fitting the curve using a least squares method.[8] The slope of the fitted straight line is the drive system Weibull slope b_s. The life at which the drive sytem reliability equals 90% on the straight line is l_{s10}. The drive system two-parameter Weibull relationship is then

$$\ell n\left(\frac{1}{R_s}\right) = \ell n\left(\frac{1}{0.9}\right)\left(\frac{l_s}{l_{s10}}\right)^{b_s} \tag{12}$$

Renewal Theory

Renewal theory adds the renewal function to the statistical tools for estimating repair. It estimates the number of replacements as a function of a component failure distribution and its life.[9-11] Renewal theory assumes the replacement of failed components when they fail. This models an unending sequence of use and repair. Aircraft drive system maintenance follows this pattern closely. The renewal function results from a sequence of statistically predicted failures.

Consider the maintenance sequence. In a given life period, any number of failures may occur. The probability of at least one failure within a given life from the start of operation is

$$F_1(l) = F(l) = \int_0^l f(x)\mathrm{d}x \tag{13}$$

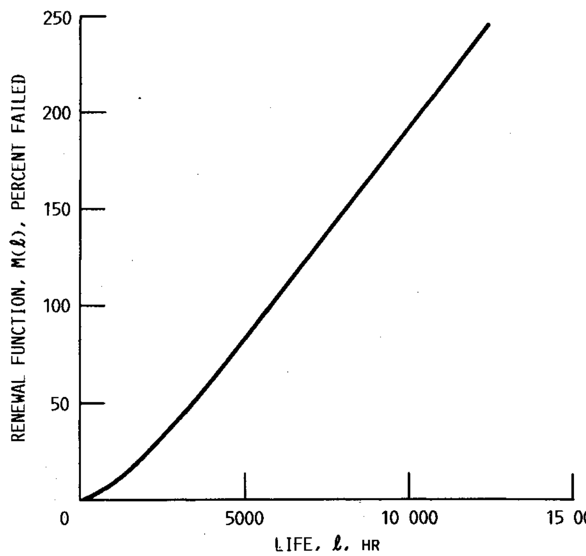

Fig. 5 Renewal function for a two-parameter Weibull distribution with Θ = 5000 h and b = 1.5.

The probability of at least two sequential failures in the period is the probability of two independent events. The first component must fail. Then a second component must begin its service life at this failure life and also fail. The probability of having at least two failures in this period is

$$F_2(l) = \int_0^l F_1(l - x)f(x)\mathrm{d}x \tag{14}$$

In Eq. (14), x is the time at which the first failure occurs. This can happen any time between zero and l. At $x = 0$, the entire life is available for the first failure probability. The probabilities of the second failure and the combination event are both zero. As x increases from zero to l, the probability of the first failure happening at time x decreases. The probability of the second failure increases. At $x = l$, the entire life is available for the second failure probability. The probability of the first failure is zero. The probability of the combined event is thus zero as well at $x = l$. The integral defines a function for the probability of at least two failures in the life period from zero to l.

Equation (14) repeats indefinitely with increasing subscripts. The probability of having at least k failures in the period from zero to l is

$$F_k(l) = \int_0^l F_{k-1}(l - x)f(x)\mathrm{d}x \tag{15}$$

In Eq. (15), $F_{k-1}(l - x)$ is the probability of having at least $(k - 1)$ failures in the period from zero to $(l - x)$. The probability of at least k failures increases as the number of failures decrease. Also, the more life available for a failure, the greater is the chance that it will occur.

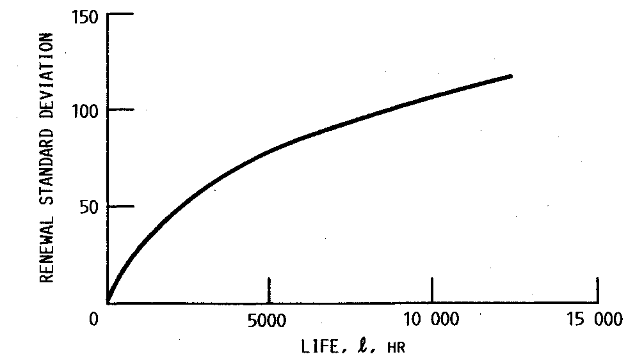

Fig. 6 Renewal function standard deviation for a two-parameter Weibull distribution with $\Theta = 5000$ h and $b = 1.5$.

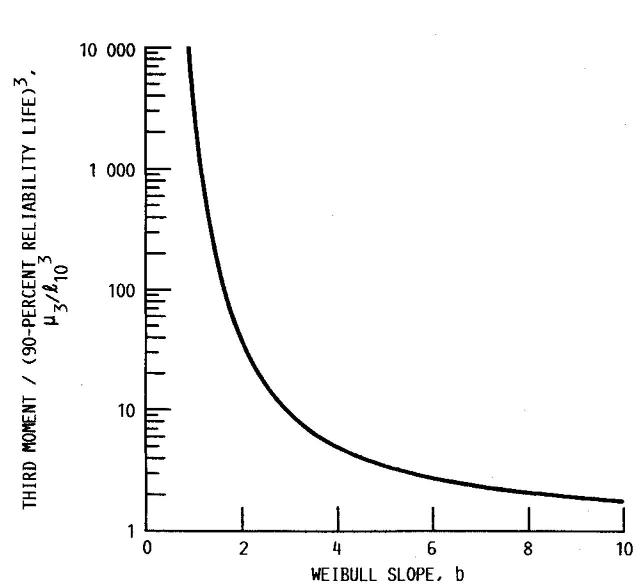

Fig. 7 Third moment to 90% reliability life ratio for a Weibull distribution as a function of the Weibull slope.

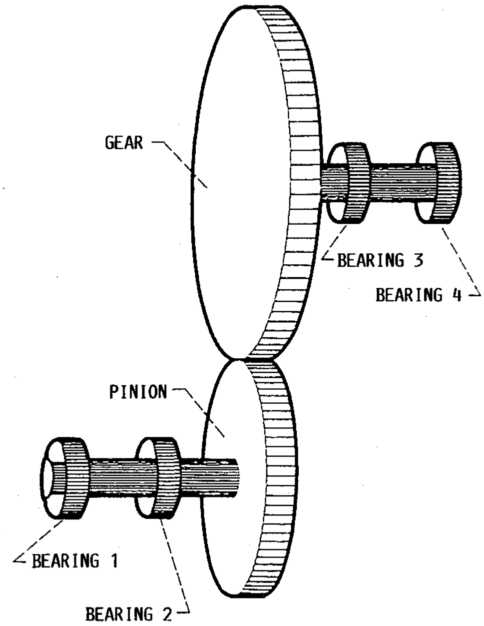

Fig. 8 Single-mesh transmission example.

The mean number of failures is the infinite sum of the probabilities of at least k failures in the life period l. This function, $M(l)$, is the renewal function. It is expressed as

$$M(l) = \sum_{k=1}^{\infty} F_k(l) \tag{16}$$

Equations (15) and (16) yield the number of replacements needed to support a maintenance schedule. The solution involves a large series of convolution integrals. The equations apply to any failure distribution. However, the solution is not easy to obtain. Figure 5 shows the renewal function for a component with a two-parameter Weibull reliability. The component life has $\Theta = 5000$ h and $b = 1.5$. Tabulated solutions to the renewal function for the two-parameter Weibull distribution are available.[11,16]

An approximation for the renewal function from Ref. 10 is

$$M_e(l) = \frac{l}{l_{\mathrm{av}}} - \frac{l_{\mathrm{av}}^2 - \sigma_f^2}{2l_{\mathrm{av}}^2} \tag{17}$$

The approximation accuracy increases as l increases. Equation (17) is an asymptote to the exact renewal function for low-scatter distributions. For high-scatter distributions it approximates the renewal function closely.

The standard deviation of the renewal function gives a measure of the scatter in replacement needs from one sample to the next. Figure 6 is a plot of the renewal function standard deviation vs life for a component which has a two-parameter Weibull reliability distribution. It has a characteristic life of 5000 h and a Weibull slope of 1.5.

The approximation for the standard deviation of the renewal function is[10]

$$\sigma_{\mathrm{me}}(l) = \left[\frac{\sigma_f^2}{l_{\mathrm{av}}^3} l + \left(\frac{l_{\mathrm{av}}^2 + \sigma_f^2}{4l_{\mathrm{av}}^4}\right)(3l_{\mathrm{av}}^2 + 5\sigma_f^2) - \frac{2\mu_3}{3l_{\mathrm{av}}^3}\right]^{1/2} \tag{18}$$

where μ_3 is the third moment of the life distribution. For the two-parameter Weibull distribution, the third moment is

$$\mu_3 = \int_0^{\infty} l^3 f(l)\mathrm{d}l = \Theta^3\Gamma\left(1 + \frac{3}{b}\right) \tag{19}$$

Table 1 Single-mesh transmission properties

	l_{10}, h	l_{av}, h	σ_f, h
Bearing 1	2640	16,200	13,560
Bearing 2	4820	29,570	24,750
Pinion	2480	5,410	2,320
Bearing 3	7230	44,360	37,130
Bearing 4	3960	24,300	20,330
Gear	3170	6,920	2,960
Transmission	1060	3,990	2,600

Table 2 Single-mesh transmission component replacement predictions using partial replacement method; $Q = 50$ aircraft, $l = 10{,}000$ h

	N_r, h	σ_{me}, h	$N_{r,90}$, h
Bearing 1	23	5.2	30
Bearing 2	9	4.1	15
Pinion	72	4.7	78
Bearing 3	4	3.6	8
Bearing 4	13	4.4	19
Gear	52	4.3	57
Total	173		207

Figure 7 is a plot of the ratio of the third moment to the 90% reliability life as a function of the Weibull slope.

Confidence Statistics

In predicting replacement rates and maintenance inventories, direct theory provides mean or average estimates. These estimates come from the statistics of a universal population (that is, an infinite number of samples). In any real situation, the number of drive systems under service is a finite sample. Confidence statistics estimate how differently a small sample may behave from its universal population.[16]

For many samples of the same size, the mean of the samples has a normal distribution about the overall mean. The standard deviation of the means is

$$\sigma_{av} = \frac{\sigma_f}{\sqrt{Q}} \tag{20}$$

The standard deviation of the number of replacements is

$$\sigma_r = \sqrt{Q}\,\sigma_{me} \tag{21}$$

Also, the total number of replacements for the sample is

$$N_r = QM(l) \tag{22}$$

In reliability predictions, the lower confidence bound is valuable in aircraft applications. The confidence distribution estimates the mean life that will be lower than the mean life of a chosen percentage of all samples of a given size. This life is less than the mean life for the entire population. For a 90% confidence,

$$l_{av,90} = l_{av} - z_{10}\sigma_{av} \tag{23}$$

where z_{10} is the number of standard deviations below the mean that cuts off 10% of the population. For a normal distribution, $z_{10} = 1.282$.[4,14] Ninety percent of the normal distribution lie above $l_{av,90}$ and 10% lie below $l_{av,90}$.

With a 90% confidence that the replacements will be less, the replacement estimate for a component from zero to life l is

$$N_{r,90} = N_r + z_{10}\sigma_r \tag{24}$$

Since the behavior of samples differs from the behavior of the "ideal" distribution, confidence estimates are helpful. With the confidence estimates, one can see the effects of sample size on the life and replacement estimates during the design phase.

Example

Consider the single-mesh transmission in Fig. 8. For this example, the 90% reliability lives for the bearings and gears are shown in Table 1. The Weibull slope for the bearings is 1.2, and for the gears is 2.5. For a fleet of $Q = 50$ aircraft, we would like to estimate the number of overhauls in the first 10,000 h of service and the number of replacement components needed to support these overhauls. Two types of overhaul are treated: 1) full replacement and 2) failed component replacement only. All estimates will be with 90% confidence for the 50-aircraft sample size. From Eq. (7) or Fig. 3, the average lives were determined for each component. From Eq. (9) or Fig. 4, the standard deviations for each component were determined. The results are shown in Table 1.

Full Replacement

We assume the aircraft transmissions have accurate onboard monitoring systems that indicate a failure and need for an overhaul. The full replacement method assumes all components of a transmission are replaced during an overhaul of a failed transmission. The two-parameter Weibull system model of Eq. (12) is used. The parameters b_s and l_{s10} for Eq. (12) come from a least-squares fit to Eq. (11). The two-parameter Weibull slope is $b_s = 1.57$ for the transmission and the system 90% reliability life is $l_{s10} = 1060$ h. From Eq. (4) or Fig. 1, the transmission characteristic life is $\Theta = 4440$ h. From Eq. (7) or Fig. 3, the transmission average life is $l_{av} = 3990$ h. From Eq. (9) or Fig. 4, the standard deviation of the transmission life is $\sigma_f = 2600$ h. Table 1 includes these results.

The renewal function can estimate the number of transmissions needed for full replacement in a continual sequence of failure warning, repair, and return to service for the 50 aircraft. For an average life of 3990 h and a standard deviation life of 2600 h, Eq. (17) gives the renewal function as

$$M_e = \frac{l}{3990} - \frac{(3990)^2 - (2660)^2}{2(3990)^2} = \frac{l}{3990} - 0.278 \tag{25}$$

From Eq. (22), the total number of replacements in the period from 0–l h is

$$N_r = 50\left[\frac{l}{3990} - 0.278\right] = \frac{l}{79.8} - 13.9 \tag{26}$$

From Eq. (19) or Fig. 7, the third moment of the transmission life distribution is $\mu_3 = 1.62 \times 10^{11}$ h^3. From Eq. (18), the standard deviation of the renewal function for the transmission is

$$\sigma_{me}(l) = \left\{\frac{(2600)^2}{(3990)^3}\,l + \left[\frac{(3990)^2 + (2600)^2}{4(3990)^4}\right]\left[3(3990)^2 + 5(2600)^2\right] - \frac{2(1.62\times 10^{11})}{3(3990)^3}\right\}^{1/2}$$

$$\sigma_{me}(l) = \left[\frac{l}{9400} + 0.124\right]^{1/2} \tag{27}$$

From Eq. (21) the standard deviation of the number of replacements in the period from 0 to l is

$$\sigma_r = \sqrt{50}\left[\frac{l}{9400} + 0.124\right]^{1/2} = \left[\frac{l}{188} + 6.22\right]^{1/2} \tag{28}$$

With a 90% confidence that the replacements will be less, the replacement estimate for complete transmissions using Eq. (24) is

$$N_{r,90} = \left[\frac{l}{79.8} - 13.9\right] + 1.282\left[\frac{l}{188} + 6.22\right]^{1/2} \tag{29}$$

For the first 10,000 h of operation, Eq. (26) estimates an average number of replacements for the 50 aircraft of 111. Including a confidence limit of 90%, Eq. (29) boosts this estimate to 121 transmission replacements for 500,000 fleet service hours. If we assume the mean time between overhauls equals the average time to a failure, this represents a mean time between overhauls of 500,000/121 or 4130 h.

Renewal theory estimates with a 90% confidence that 121 transmissions will be needed for 121 overhauls for the 50 aircraft in the first 10,000 h of service per aircraft. Also, 726 components are required since all six transmission components are replaced at the overhauls. In fleet service hours, the mean time between overhauls is estimated to be 4130 h with a 90% confidence. The high service reliability of the aircraft transmissions is provided by the onboard monitoring system that calls for the overhauls. These estimates are for cost and scheduling purposes.

Partial Replacement

The partial replacement method assumes only the failed components of a transmission are replaced during an overhaul. Again we assume the aircraft transmissions have accurate onboard monitoring systems that indicate a component failure. The renewal function can estimate the number of replacement components needed, also. By repeating the calculations of Eqs. (25–29) for each of the six components in the transmission, one can estimate the number of components needed to support a partial repair maintenance schedule.

Table 2 summarizes these calculations for the four bearings and two gears in the transmission. Adding the total number of components that renewal theory estimates will need replacement yields 72 bearings and 135 gears for a total of 207 components. If each component failure required its own overhaul, then 207 overhauls would be required with the same 90% confidence as used for the full replacement calculations. Dividing the 500,000 fleet service hours by the maximum number of 207 overhauls yields an estimate for the mean time between overhauls equal to 2420 h. This is 1710 h less than the mean time between overhauls for full transmission replacement. By only replacing the failed components, 86 more overhauls would result but 519 fewer parts would be required. The same high reliability would be present for both maintenance procedures due to the onboard failure monitoring system.

Summary of Results

Two methods of estimating the time between transmission overhauls and the number of replacement components needed were presented. The first method assumes replacement of all components during an overhaul of a failed transmission (full replacement method). The second method assumes replacement of failed components only (partial replacement method). Both methods assume the transmission components follow a two-parameter Weibull failure distribution. Formulas for the mean and standard deviation of the two-parameter Weibull distribution were given.

Renewal theory was presented to estimate the number of component replacements in a transmission for both methods. Approximation formulas valid for a two-parameter Weibull distribution were given for the mean and standard deviation of the renewal function. For the partial replacement method, renewal theory was used with the individual component life predictions to estimate the number of component replacements needed and the transmission time between overhauls. For the full replacement method, renewal theory was used with a transmission system life model to estimate the number of replacement transmissions needed and the transmission time between overhauls. The system life model was based on a two-parameter Weibull distribution. The relationship between the system life model and the component life models was presented.

Confidence statistics were applied to both methods to improve the statistical estimate of sample behavior. Single-sided confidence theory was presented for both the number of replacements and the time between overhaul estimates. A transmission example was presented to illustrate use of the methods. A comparison of time between overhaul and spare part requirements was made in the example between full transmission replacement and partial component replacement overhauls.

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